§5.5: Distribution of a Linear Combination

Suppose we have n (maybe not IID) jointly distributed random variables

$$X_1, X_2, X_3, \ldots, X_n$$

The statistic

$$Y = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

a's are numerical constants.

is a "linear combination" of the X_k

Example: If all $a_k = 1$, then the linear combination is $Y = T_o = X_1 + X_2 + \cdots + X_n$

Example: If all $a_k = \frac{1}{n}$, then the linear combination is $Y = \bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n$

Example: If $a_1 = 1$ and $a_2 = -1$ then we have the difference $Y = X_1 - X_2$

Generalizing work in §5.2 we get the following:

- Expected value is linear

$$E[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n] = a_1 E[X_1] + a_2 E[X_2] + \cdots + a_n E[X_n]$$

- If (X_k) are independent then

$$\text{Var}[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n] = a_1^2 \text{Var}[X_1] + a_2^2 \text{Var}[X_2] + \cdots + a_n^2 \text{Var}[X_n]$$

- If (X_k) are not independent then

$$\text{Var}[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n] = \sum_{i,j} a_i a_j \text{Cov}[X_i, X_j]$$

note: $\text{Cov}[X_k, X_k] = \text{Var}[X_k]$

The "Little Theorem" from §5.4 also generalizes:

> If $X_1, \ldots, X_n$ are all Normal random variables then so is $Y = a_1 X_1 + \cdots + a_n X_n$

In particular, note that

$$E[X_1 - X_2] = E[X_1] - E[X_2]$$

BUT (slightly confusingly)

$$Var[X_1 - X_2] = Var[X_1] + Var[X_2]$$

If X_1, X_2 are IID Normal(μ, σ)

then

$$X_1 - X_2 \sim \text{Normal}(0, \underline{\sqrt{2}\sigma})$$

(Because $Var[X_1 - X_2] = 2\,Var[X]$
$= 2\sigma^2$)

[Be careful! Variances (and standard deviations) **NEVER** subtract!!]

Example: Suppose X_1 & X_2 are IID Normal with $\mu = 10$ and $\sigma = 4$. What is probability that $X_1 - X_2 > 2$?

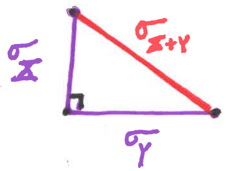
$$X_1, X_2 \sim \text{Normal}(10, 4)$$

so $(X_1 - X_2) \sim \text{Normal}(0, 4\sqrt{2})$

$$4\sqrt{2} \leftarrow \sqrt{4^2 + 4^2}$$

$$P(X_1 - X_2 > 2) = 1 - P(X_1 - X_2 < 2)$$
$$= 1 - \text{pnorm}(2, 0, 4\sqrt{2})$$
$$\approx 0.36$$

Note: Standard Deviations of independent r.v.s add like the Pythagorean Theorem



$$\boxed{\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2}$$

(If not independent then is $Cov[X, Y]$ like the "cos θ" in Law of Cosines?)